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Evaluation of a New Estimator

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ABSTRACT

An estimator is used to estimate the unknown parameter in a linear regression model. In this paper, a new estimator was derived from further modification of the Liu-type Estimator. The performance of the new estimator was evaluated by comparing its mean squared error with the mean squared errors of other estimators. It was found that there is a reduction in mean squared error in the new estimator under certain conditions.

Keywords: Estimator, mean squared error, regression

INTRODUCTION

A linear regression model is used to describe the relationship between a dependent variable and one or several independent variables. A linear regression model is generally written as $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_p x_p + \varepsilon$, where β_j, $j = 0, 1, 2, ..., p$, is a parameter and ε is the error term. The parameter β_j in the linear regression model is unknown and is to be estimated from data. There are many existing estimators in regression analysis such as the Ordinary Least Squares Estimator, the Shrunken Estimator (Stein, 1960; cited by Hocking *et al.*, 1976), the Ordinary Ridge Regression Estimator (Hoerl and Kennard, 1970), the $r - k$ Class Estimator (Baye and Parker, 1984), the Liu Estimator (Liu, 1993), the $r - d$ Class Estimator (Kaciranlar and Sakallioglu, 2001) and the Liu-type Estimator (Liu, 2003; Liu, 2004).

In this paper, a new estimator is developed to improve the accuracy of parameter estimates in regression analysis. The new estimator is developed by modification of the Liu-type Estimator. Its performance is evaluated by comparing it with other estimators. The Ordinary Ridge Regression Estimator and the Liu Estimator obtained much interest from the researchers as many studies have been done on these estimators (Baye and Parker, 1984; Pliskin, 1987; Sarkar, 1996; Kaciranlar *et al.*, 1998; Kaciranlar *et al.*, 1999; Sakallioglu *et al.*, 2001; Kaciranlar and Sakallioglu, 2001). In this paper, the performance of the new estimator is evaluated by comparing its mean squared error with the mean squared errors of the Ordinary Ridge Regression Estimator and the Liu Estimator.

A NEW ESTIMATOR

Suppose a linear regression model with standardized variables can be written in the matrix form (Akdeniz and Erol, 2003)

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$$Y = Z\gamma + \varepsilon, \qquad [2.1]$$

where Y is a vector of standardized dependent variables, Z is a matrix of standardized independent variables, γ is a vector of parameters, and ε is a vector of errors such that $\varepsilon \sim N(0, \sigma^2 I)$.

Then, the linear regression model, $Y = Z\gamma + \varepsilon$, can be transformed into a canonical form (Akdeniz and Erol, 2003)

$$Y = X\beta + \varepsilon \qquad [2.2]$$

where $X = ZT$, $\beta = T`\gamma$ is a vector of parameters, r is a vector of parameters in the regression model $Y = Z\gamma + \varepsilon$, $X`X = \lambda$, T is an orthonormal matrix consisting of the eigenvectors of $Z'Z$ and λ is $= \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ a diagonal matrix whose diagonal elements are the eigenvalues of $Z'Z$.

In this paper, a new estimator is introduced from further modification of the Liu-type Estimator. Here, a special case of Liu-type Estimator (Liu, 2003) is considered:

$$\hat{\beta}_c = (X'X + cI)^{-1}\ (X'Y + \hat{\beta})$$

$$= [I - (\lambda + cI)^{-1}\ (c-1)\ \hat{\beta} \qquad [2.3]$$

where c is the biasing parameter that is added to the diagonal of matrix X'X, $c > 1$, and $\hat{\beta} = (X'X)^{-1}X'Y$ is the Ordinary Least Squares Estimator of parameter β.

The bias and the mean squared error of $\hat{\beta}_c$ are given by Equations [2.4] and [2.5], respectively.

$$\mathrm{bias}(\hat{\beta}_c) = -\ (\lambda + cI)^{-1}\ (c-1)\beta \qquad [2.4]$$

$$\mathrm{mse}(\hat{\beta}_c) = \sum_{j=1}^{p}\left[\frac{(\lambda_j+1)^2\sigma^2}{\lambda_j(\lambda_j+c)^2} + \frac{(c-1)^2\beta_j^2}{(\lambda_j+c)^2}\right] \qquad [2.5]$$

The estimator, $\hat{\beta}_c$, is a biased estimator since there is a certain amount of bias in the estimator. A new estimator is introduced by reducing the bias in $\hat{\beta}_c$. Let $\mathrm{bias}(\hat{\beta}_c)$ be the bias of $\hat{\beta}$ with the unknown β replaced by $\hat{\beta}$. Thus, $\mathrm{bias}(\hat{\beta}_c)$ is given by

$$\mathrm{bias}(\hat{\beta}) = -(\lambda + cI)^{-1}\ (c-1)\ (c-1)\,\hat{\beta} \qquad [2.6]$$

Hence, the new estimator, $\tilde{\beta}_c$, is given by

$$\begin{aligned} \tilde{\beta}_c &= \hat{\beta} - \mathrm{bias}\ (\hat{\beta}_c) \\ &= \hat{\beta} - [-(\lambda + cI)^{-1}\ (c-1)\,\hat{\beta}] \\ &= \hat{\beta} + (\lambda + cI)^{-1}\ (c-1)\,\hat{\beta}_c \end{aligned}$$

$$= [\mathrm{I} + (\lambda + d)^{-1}\ (c-1)]\,\hat{\beta}_c$$
$$= [\mathrm{I} + (\lambda + d)^{-1}\ (c-1)][\mathrm{I} - (\lambda + d)^{-1})(c-1)\,\hat{\beta}_c$$
$$= [\mathrm{I} + (\lambda + d)^{-2})(c-1)^2]\,\hat{\beta}_c \qquad [2.7]$$

where $c > 1$.

The bias, variance-covariance matrix and mean squared error of the new estimator are given by Equations [2.8], [2.9] and [2.10], respectively.

$$\begin{aligned}\mathrm{bias}(\tilde{\beta}_c) &= E(\tilde{\beta}_c) - \beta \\ &= [\mathrm{I} - (\lambda + d)^{-2}\ (c-1)^2]\ \mathrm{E}(\hat{\beta}) - \beta \\ &= [\mathrm{I} - (\lambda + d)^{-2}\ (c-1)^2]\ \beta - \beta \\ &= -\ (\lambda + d)^{-2}\ (c-1)^2\,\beta \end{aligned} \qquad [2.8]$$

$$\begin{aligned}\mathrm{Var}\ (\tilde{\beta}_c) &= \mathrm{Var}\ ([\mathrm{I} - (\lambda + d)^{-2}\ (c-1)^2]\,\hat{\beta}) \\ &= [\mathrm{I} - (\lambda + d)^{-2}\ (c-1)^2]^2\ \mathrm{Var}(\hat{\beta}) \\ &= [\mathrm{I} - (\lambda + d)^{-2}\ (c-1)^2]^2\ \sigma^2\ \lambda^{-1} \\ &= \sigma^2\,[\mathrm{I} - (\lambda + d)^{-2}\ (c-1)^2]^2 \end{aligned} \qquad [2.9]$$

$$\begin{aligned}\mathrm{mse}(\tilde{\beta}_c) &= \mathrm{Var}(\tilde{\beta}_c) + [\mathrm{bias}(\tilde{\beta}_c)]^2 \\ &= \sum_{j=1}^{p}\left\{\left[1-\left(\frac{c-1}{\lambda_j+c}\right)^2\right]^2\frac{\sigma^2}{\lambda_j}\right\} + \sum_{j=1}^{p}\left[-\left(\frac{c-1}{\lambda_j+c}\right)^2\beta_j\right]^2 \\ &= \sum_{j=1}^{p}\left\{\left[1-\left(\frac{c-1}{\lambda_j+c}\right)^2\right]^2\frac{\sigma^2}{\lambda_j} + \left(\frac{c-1}{\lambda_j+c}\right)^4\beta_j^2\right\} \end{aligned} \qquad [2.10]$$

It is found that the new estimator, $\tilde{\beta}_c$, has a reduction in bias compared to the special case of Liu-type Estimator, $\hat{\beta}_c$. This can be seen by considering the difference of the bias between these two estimators in terms of their individual magnitude:

$$\begin{aligned}\left|\mathrm{bias}((\tilde{\beta}_c)_j)\right| - \left|\mathrm{bias}((\hat{\beta}_c)_j)\right| &\left|-\left(\frac{c-1}{\lambda_j+c}\right)^2\beta_j\right| - \left|-\left(\frac{c-1}{\lambda_j+c}\right)\beta_j\right| \\ &= \left(\frac{c-1}{\lambda_j+c}\right)^2|\beta_j| - \left(\frac{c-1}{\lambda_j+c}\right)|\beta_j| \\ &= \left(\frac{c-1}{\lambda_j+c} - 1\right)\left(\frac{c-1}{\lambda_j+c}\right)|\beta_j| \\ &= -\left(\frac{\lambda_j+1}{\lambda_j+c}\right)\left(\frac{c-1}{\lambda_j+c}\right)|\beta_j| \end{aligned} \qquad [2.11]$$

$j = 1, 2, \ldots, p$.

Since $c > 1$, thus $\left|\text{bias}((\tilde{\beta}_c)_j)\right| - \left|\text{bias}((\hat{\beta}_c)_j)\right| < 0$. This implies that the magnitude of the bias of $\tilde{\beta}_c$ is less than magnitude of the bias of $\hat{\beta}_c$.

MEAN SQUARED ERROR AS AN EVALUATION TOOL FOR ESTIMATORS

The performance of the new estimator is further evaluated by comparing its mean squared error with other estimators. In general, the mean squared error of an estimator is used as a measure of the goodness of the estimator. Let $\text{mse}(\tilde{\beta})$ denote the mean squared error of an estimator, $\tilde{\beta}$. Then, the $\text{mse}(\tilde{\beta})$ is given by

$$\text{mse}(\tilde{\beta}) = E\left[(\tilde{\beta}-\beta)'(\tilde{\beta}-\beta)\right]. \qquad [3.1]$$

Suppose $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are two estimators of the parameter β. The estimator $\tilde{\beta}_2$ is superior to the estimator $\tilde{\beta}_1$ if the mean squared error of $\tilde{\beta}_2$ is smaller than the mean squared error of $\tilde{\beta}_1$, that is, Suppose the estimator, $\tilde{\beta}_1$, can be represented as a product of a matrix A_1 and the vector of standardized dependent variables, Y, that is, $\tilde{\beta}_1 = A_1 Y$. Suppose also that the estimator, $\tilde{\beta}_2$, can be represented as a product of a matrix A_2 and the vector of standardized dependent variables, Y, that is, $\tilde{\beta}_2 = A_2 Y$. Let $\text{bias}(\tilde{\beta}_1)$ and $\text{bias}(\tilde{\beta}_2)$ denote the bias of the estimators, $\tilde{\beta}_1$ and $\tilde{\beta}_2$ respectively. Then, the condition for $\text{mse}(\tilde{\beta}_1) > \text{mse}(\tilde{\beta}_2)$ is given by Theorem 3.1.

Theorem 3.1. The conditions for $\text{mse}(\tilde{\beta}_1) > \text{mse}(\tilde{\beta}_2)$ are:

(a) $A_1A_1' - A_2A_2'$ is a positive definite matrix, and

(b) $\left[\text{bias}(\tilde{\beta}_2)\right]\left(A_1A_1' - A_2A_2'\right)^{-1}\left[\text{bias}(\tilde{\beta}_2)\right] < \sigma^2$.

Proof. Let $M(\tilde{\beta}_1)$ and $M(\tilde{\beta}_2)$ denote the mean squared error matrices of the estimators, $\tilde{\beta}_1$ and $\tilde{\beta}_2$, respectively. Let $\text{Var}(\tilde{\beta}_1)$ and $\text{Var}(\tilde{\beta}_2)$ denote the variance-covariance matrices of the estimators, $\tilde{\beta}_1$ and $\tilde{\beta}_2$, respectively. The mean squared error of an estimator is equal to the variance of the estimator plus the square of its bias. Hence, the mean squared error matrices, $M(\tilde{\beta}_1)$ and $M(\tilde{\beta}_2)$, are given by Equations [3.2] and [3.3], respectively.

$$\begin{aligned} M(\tilde{\beta}_1) &= \text{Var}(\tilde{\beta}_1) + \left[\text{bias}(\tilde{\beta}_1)\right]\left[\text{bias}(\tilde{\beta}_1)\right]' \\ &= \text{Var}(A_1 Y) + \left[\text{bias}(\tilde{\beta}_1)\right]\left[\text{bias}(\tilde{\beta}_1)\right]' \\ &= \sigma^2 A_1A_2' + \left[\text{bias}(\tilde{\beta}_1)\right]\left[\text{bias}(\tilde{\beta}_1)\right]' \end{aligned} \qquad [3.2]$$

$$M(\tilde{\beta}_2) = \sigma^2 A_2A_2' \left[\text{bias}(\tilde{\beta}_2)\right]\left[\text{bias}(\tilde{\beta}_2)\right]' \qquad [3.3]$$

Thus, $M(\tilde{\beta}_1) - M(\tilde{\beta}_2)$ is given by
$M(\tilde{\beta}_1) - M(\tilde{\beta}_2)$

$$= \left\{\sigma\left(A_1A_1' - A_2A_2'\right) - \left[\text{bias}(\tilde{\beta}_2)\right]\left[\text{bias}(\tilde{\beta}_2)\right]]\right\} + \left[\text{bias}(\tilde{\beta}_1)\right]\left[\text{bias}(\tilde{\beta}_1)\right]. \qquad [3.4]$$

Note that $\left[\text{bias}(\tilde{\beta}_1)\right]\left[\text{bias}(\tilde{\beta}_1)\right]'$ is positive definite. Therefore, $M(\tilde{\beta}_1) - M(\tilde{\beta}_2)$ is positive definite if $\sigma_2\ (A_1A_1' - A_2A_2') - \left[\text{bias}(\tilde{\beta}_2)\right]\left[\text{bias}(\tilde{\beta}_2)\right]'$ is also positive definite.

Applying the theorem from Farebrother (1976): Let Q be a $p \times p$ positive definite matrix, ψ a nonzero $p \times 1$ vector and ς a positive scalar, then $\varsigma Q - \psi\psi'$ is positive definite if and only if $\psi' Q^{-1} \psi < \varsigma$. Thus, the conditions for $\sigma_2\ (A_1A_1' - A_2A_2') - \left[\text{bias}(\tilde{\beta}_2)\right]\left[\text{bias}(\tilde{\beta}_2)\right]'$ to be positive definite are:

(a) $A_1A_1' - A_2A_2'$ is a positive definite matrix and

(b) $\left[\text{bias}(\tilde{\beta}_2)\right]\ (A_1A_1' - A_2A_2')^{-1}\ \left[\text{bias}(\tilde{\beta}_2)\right] < \sigma^2$.

On the other hand, Theobald (1974) considered a weighted sum of the coefficient mean squared error as another measure of the goodness of an estimator. The weighted sum of the coefficient mean squared error is known as the generalized mean squared error (Sakallioglu *et al.*, 2001). The generalized mean squared error of an estimator, $\tilde{\beta}$, is given by

$$\text{gmse}(\tilde{\beta}) = E\left[(\tilde{\beta}-\beta)'(\tilde{\beta}-\beta)\right], \tag{3.5}$$

where B is a nonnegative definite matrix.

Theobald (1974) established a relationship between the generalized mean squared error and the mean squared error matrix of an estimator: Suppose there are two estimators, namely, $\tilde{\beta}_1$ and $\tilde{\beta}_2$, the following conditions are equivalent:

(a) $M(\tilde{\beta}_1) - M(\tilde{\beta}_2)$ is positive definite, and

(b) $\text{gmse}(\tilde{\beta}_1) > \text{gmse}(\tilde{\beta}_2)$ for all positive definite matrix B.

Note that $\text{gmse}(\tilde{\beta})$ is equal to msegmse ($\tilde{\beta}$) when the matrix B in Equation [3.5] is equal to an identity matrix I. From this, we see that the condition for $M(\tilde{\beta}_1) - M(\tilde{\beta}_2)$ is positive definite is equivalent to the condition for $\text{mse}(\tilde{\beta}_1) - \text{mse}(\tilde{\beta}_2)$.
Thus, the conditions for $\text{mse}(\tilde{\beta}_1) - \text{mse}(\tilde{\beta}_2)$ are:

(a) $A_1A_1' - A_2A_2'$ is a positive definite matrix and

(b) $\left[\text{bias}(\tilde{\beta}_2)\right]'\ (A_1A_1' - A_2A_2')^{-1}\ \left[\text{bias}(\tilde{\beta}_2)\right] < \sigma^2$.

Hence, the proof for Theorem 3.1 is completed. ❑

The comparison between the new estimator and other estimators is performed by applying the concept of Theorem 3.1.

THE EVALUATION OF THE NEW ESTIMATOR

The Ordinary Ridge Regression Estimator and the Liu Estimator are two biased estimators. The new estimator is compared with these two estimators in terms of mean squared error.

The Ordinary Ridge Regression Estimator is given by (Hoerl and Kennard, 1970)

$$\begin{aligned}\hat{\beta}_k &= (X'X + kI)^{-1}\ X'Y \\ &= A_k\ Y,\end{aligned} \tag{4.1}$$

where $A_k = (X'X + kI)^{-1} X'$, $k > 0$.
Hence, $A_k A_k'$ is given by (Hoerl and Kennard, 1970)

$$A_k A_k' = (\pi + kI)^{-2} \lambda . \tag{4.2}$$

The bias and the mean squared error of $\hat{\beta}_k$ are given by Equations [4.3] and [4.4], respectively (Hoerl and Kennard, 1970).

$$\text{bias}(\hat{\beta}_k) = -(X'X + kI)^{-1} k\beta \tag{4.3}$$

$$\text{mse}(\hat{\beta}_k) = \sum_{j=1}^{p} \left[\frac{\lambda_j \sigma^2}{(\lambda_j + k)^2} + \frac{k^2 \beta_j^2}{(\lambda_j + k)^2} \right] \tag{4.4}$$

The Liu Estimator is given by (Liu, 1993)

$$\begin{aligned} \hat{\beta}_d &= (X'X + I)^{-1} (X'X + dI)(X'X)^{-1} X'Y \\ &= A_d Y, \end{aligned} \tag{4.5}$$

where $A_d = (X'X + I)^{-1} (X'X + dI)(X'X)^{-1} X'$, $0 < d < 1$.
Hence, $A_d A_d'$ is given by (Liu, 1993)

$$A_d A_d' = (\lambda + I)^{-2} (\lambda + dI)^2 \lambda^{-1} \tag{4.6}$$

The bias and the mean squared error of $\hat{\beta}_d$ are given by Equations [4.7] and [4.8], respectively (Liu, 1993).

$$\text{bias}(\hat{\beta}_d) = -(X'X + I)^{-1} (1 - d)\beta \tag{4.7}$$

$$\text{mse}(\hat{\beta}_d) = \sum_{j=1}^{p} \left[\frac{(\lambda_j + d)^2 \sigma^2}{\lambda_j (\lambda_j + 1)^2} + \frac{(1-d)^2 \beta_j^2}{(\lambda_j + 1)^2} \right] \tag{4.8}$$

The new estimator, $(\tilde{\beta}_c)$, can also be written as

$$\begin{aligned} (\tilde{\beta}_c) &= \left[I - (\lambda + dI)^{-2} (c - 1)^2 \right] \hat{\beta} \\ &= \left[I - (\lambda + dI)^{-2} (c - 1)^2 \right] \lambda^{-1} X'Y \\ &= A_c Y, \end{aligned} \tag{4.9}$$

where $A_c = \left[I - (\lambda + dI)^{-2} (c - 1)^2 \right] \lambda^{-1} X'$, $c > 1$.
Hence, $A_c A_c'$, is given by

$$A_c A_c' = [I - (\lambda + dI)^{-2} (c - 1)^2]^2 \lambda^{-1}. \tag{4.10}$$

Theorem 4.1 shows the comparison between $\tilde{\beta}_c$ and $\hat{\beta}_k$ while Theorem 4.2 shows the comparison between $\tilde{\beta}_c$ and $\hat{\beta}_d$.

Theorem 4.1. Let c in $\tilde{\beta}_c$ be fixed and $c > 1$.

(a) If $[\text{bias}(\tilde{\beta}_c)]'(A_kA_k' - A_cA_c')^{-1}[\text{bias}(\tilde{\beta}_c) < \sigma^2$, then $\text{mse}(\hat{\beta}_k) > \text{mse}(\tilde{\beta}_c)$ for $0 < k < \min\{(b_1)_j\}$, and

(b) If $[\text{bias}(\tilde{\beta}_k)]'(A_cA_c' - A_kA_k')^{-1}[\text{bias}(\hat{\beta}_k) < \sigma^2$, then $\text{mse}(\tilde{\beta}_c) > \text{mse}(\hat{\beta}_k)$ for $0 < \max\{(b_1)_j\} < k$,

where $(b_i)_j = \dfrac{\lambda_j(c-1)^2}{\lambda_j^2 + 2c\lambda_j + 2c - 1}$, $j = 1, 2, \ldots, p$.

Proof.

(a) From Theorem 3.1, the conditions for $\text{mse}(\hat{\beta}_k) > \text{mse}(\tilde{\beta}_c)$ are:

(i) $A_kA_k' - A_cA_c'$ is a positive definite matrix, and

(ii) $[\text{bias}(\tilde{\beta}_c)]'(A_kA_k' - A_cA_c')^{-1}[\text{bias}(\tilde{\beta}_c)] < \sigma^2$.

Using $A_kA_k' = (\lambda + kI)^{-2}\lambda$ (Equation [4.2]), and $A_cA_c' = [I - (\lambda + cI)^{-2}(c-1)^2]^2\lambda^{-1}$ (Equation [4.10]), the matrix $A_kA_k' - A_cA_c'$ is a $p \times p$ diagonal matrix with diagonal elements $\dfrac{\lambda_j}{(\lambda_j + k)^2} - \dfrac{1}{\lambda_j}\left[1 - \dfrac{(c-1)^2}{(\lambda_j + c)^2}\right]^2$, $j = 1, 2, \ldots, p$. Hence, $A_kA_k' - A_cA_c'$ is a positive definite matrix if and only if

$$\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{1}{\lambda_j}\left[1 - \frac{(c-1)^2}{(\lambda_j + c)^2}\right]^2 > 0$$

$$\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{1}{\lambda_j}\left[\frac{(\lambda_j + c)^2 - (c-1)^2}{(\lambda_j + c)^2}\right]^2 > 0$$

$$\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{(\lambda_j + 2c - 1)^2(\lambda_j + 1)^2}{\lambda_j(\lambda_j + c)^4} > 0$$

$$\frac{\lambda_j^2(\lambda_j + c)^4 - (\lambda_j + k)^2(\lambda_j + 2c - 1)^2(\lambda_j + 1)^2}{\lambda_j(\lambda_j + k)^2(\lambda_j + c)^4} > 0$$

$$\lambda_j^2(\lambda_j + c)^4 - (\lambda_j + k)^2(\lambda_j + 2c - 1)^2(\lambda_j + 1)^2 > 0$$

$$\lambda_j(\lambda_j + c)^2 - (\lambda_j + k)(\lambda_j + 2c - 1)(\lambda_j + 1) > 0$$

$$\lambda_j(c-1)^2 - k(\lambda_j^2 + 2c\lambda_j + 2c - 1) > 0$$

$$k < \frac{\lambda_j(c-1)^2}{\lambda_j^2 + 2c\lambda_j + 2c - 1}.$$

Let $(b_1)_j = \dfrac{\lambda_j(c-1)^2}{\lambda_j^2 + 2c\lambda_j + 2c - 1}$, $j = 1, 2, ..., p$. Note that $(b_1) > 0$ and $c > 1$. Hence, $A_kA_k' - A_cA_c'$ is a positive definite matrix if and only if $0 < k < \min\{(b_1)_j\}$. From Theorem 3.1, the proof for Theorem 4.1(a) is completed.

(b) From Theorem 3.1, the conditions for $\text{mse}(\tilde{\beta}_c) > \text{mse}(\hat{\beta}_k)$ are:
 (i) $A_cA_c' - A_kA_k'$ is a positive definite matrix, and
 (ii) $\left[\text{bias}(\hat{\beta}_k)\right]' (A_cA_c' - A_kA_k')^{-1} \left[\text{bias}(\hat{\beta}_k)\right] < \sigma^2$.

Hence, $A_cA_c' - A_kA_k'$ is a positive definite matrix if and only if

$$\frac{1}{\lambda_j}\left[1 - \frac{(c-1)^2}{(\lambda_j + c)^2}\right]^2 - \frac{\lambda_j}{(\lambda_j + k)^2} > 0$$

$$k > \frac{\lambda_j(c-1)^2}{\lambda_j^2 + 2c\lambda_j + 2c - 1}.$$

Let $(b_1)_j = \dfrac{\lambda_j(c-1)^2}{\lambda_j^2 + 2c\lambda_j + 2c - 1}$, $j = 1, 2, ..., P$. Hence, $A_cA_c' - A_kA_k'$ is a positive definite matrix if and only if $0 < \max\{(b_1)_j\} < k$. From Theorem 3.1, the proof for Theorem 4.1(b) is completed. ❑

Theorem 4.2. Let c in $\tilde{\beta}_c$ be fixed and $1 < c < \min\left\{\dfrac{2\lambda_j + 1 + (\lambda_j + 1)^{\frac{3}{2}}}{\lambda_j}\right\}$.

(a) If $\left[\text{bias}(\tilde{\beta}_c)\right]' (A_dA_d' - A_cA_c')^{-1} \left[\text{bias}(\tilde{\beta}_c)\right] < \sigma^2$, then $\text{mse}(\hat{\beta}_d) > \text{mse}(\tilde{\beta}_c)$ for $0 <$ max $\max\{(b_2)_j\} < d < 1$, and
(b) If $\left[\text{bias}(\hat{\beta}_d)\right]' (A_cA_c' - A_dA_d')^{-1} \left[\text{bias}(\hat{\beta}_d)\right] < \sigma^2$, then $\text{mse}(\tilde{\beta}_c) > \text{mse}(\hat{\beta}_d)$ for $0 < d < \min\{(b_2)_j\} < 1$,

where $(b_2)_j = \dfrac{\lambda_j^2 + 4\lambda_j c - \lambda_j + 2c - 1 - \lambda_j c^2}{(\lambda_j + c)^2}$, $j = 1, 2, ..., p$.

Proof.

(a) From Theorem 3.1, the conditions for $\text{mse}(\hat{\beta}_d) > \text{mse}(\hat{\beta}_c)$ are:
 (i) $A_dA_d' - A_cA_c'$ is a positive definite matrix, and
 (ii) $\left[\text{bias}(\tilde{\beta}_c)\right]' (A_dA_d' - A_cA_c')^{-1} \left[\text{bias}(\tilde{\beta}_c)\right] < \sigma^2$.

Using $A_dA_d' = (\lambda + I)^{-2} (\lambda + dI)^2 \lambda^{-1}$ (Equation [4.6]), and $A_cA_c' = \left[I - (\lambda + c)^{-2} (c-1)^2\right]^2)$ λ^{-1} (Equation [4.10]), the matrix $A_dA_d' - A_cA_c'$ is a $p \times p$ diagonal matrix with diagonal elements $\dfrac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} - \dfrac{1}{\lambda_j}\left[1 - \dfrac{(c-1)^2}{(\lambda_j + c)^2}\right]^2$, $j = 1, 2, ..., p$. Hence, $A_dA_d' - A_cA_c'$ is a positive definite matrix if and only if

$$\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2}-\frac{1}{\lambda_j}\left[1-\frac{(c-1)^2}{(\lambda_j+c)^2}\right]^2>0$$

$$\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2}-\frac{1}{\lambda_j}\left[\frac{(\lambda_j+c)^2-(c-1)^2}{(\lambda_j+c)^2}\right]^2>0$$

$$\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2}-\frac{(\lambda_j+2c-1)^2(\lambda_j+1)^2}{\lambda_j(\lambda_j+c)^4}>0$$

$$\frac{(\lambda_j+d)^2(\lambda_j+c)^4-(\lambda_j+1)^4(\lambda_j+2c-1)^2}{\lambda_j(\lambda_j+1)^2(\lambda_j+c)^4}>0$$

$$(\lambda_j+d)^2(\lambda_j+c)^4-(\lambda_j+1)^4(\lambda_j+2c-1)^2>0$$

$$(\lambda_j+d)(\lambda_j+c)^2-(\lambda_j+1)^2(\lambda_j+2c-1)>0$$

$$d(\lambda_j+c)^2-(\lambda_j^2+4\lambda_jc-\lambda_j+2c-1-\lambda_jc^2)>0$$

$$d>\frac{\lambda_j^2+4\lambda_jc-\lambda_j+2c-1-\lambda_jc^2}{(\lambda_j+c)^2}.$$

Let $(b_2)_j=\dfrac{\lambda_j^2+4\lambda_jc-\lambda_j+2c-1-\lambda_jc^2}{(\lambda_j+c)^2}$, $j = 1, 2, ..., p$. Note that d is the biasing factor of $(\hat{\beta}_d)$, where $0 < d < 1$. Thus, $0<(b_2)_j=\dfrac{\lambda_j^2+4\lambda_jc-\lambda_j+2c-1-\lambda_jc^2}{(\lambda_j+c)^2}<1$. Solving the inequality $0<\dfrac{\lambda_j^2+4\lambda_jc-\lambda_j+2c-1-\lambda_jc^2}{(\lambda_j+c)^2}<1$ for c we get $1<c<\dfrac{2\lambda_j+1+(\lambda_j+1)^{\frac{3}{2}}}{\lambda_j}$. Hence, by letting c in $\tilde{\beta}_c$ be fixed and $1 < c <$ min, $1<c<\min\left\{\dfrac{2\lambda_j+1+(\lambda_j+1)^{\frac{3}{2}}}{\lambda_j}\right\}$, $A_dA_d' - A_cA_c'$ is a positive definite matrix if and only if $0 < \max\{(b_2)_j\} < d < 1$. From Theorem 3.1, the proof for Theorem 4.2(a) is completed.

(b) From Theorem 3.1, the conditions for $\text{mse}(\tilde{\beta}_c) > \text{mse}(\hat{\beta}_d)$ are:

(i) $A_cA_c' - A_dA_d'$ is a positive definite matrix, and

(ii) $[\text{bias}(\hat{\beta}_d)]'(A_cA_c' - A_dA_d')^{-1}[\text{bias}(\hat{\beta}_d)] < \sigma^2$.

Hence, $A_cA_c' - A_dA_d'$ is a positive definite matrix if and only if

$$\frac{1}{\lambda_j}\left[1-\frac{(c-1)^2}{(\lambda_j+c)^2}\right]^2-\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2}>0$$

$$d<\frac{\lambda_j^2+4\lambda_jc-\lambda_j+2c-1-\lambda_jc^2}{(\lambda_j+c)^2}.$$

Let $(b_2)_j = \dfrac{\lambda_j^2 + 4\lambda_j c - \lambda_j + 2c - 1 - \lambda_j c^2}{(\lambda_j + c)^2}$, $j = 1, 2, ..., p$. By letting c in $\hat{\beta}_c$ be fixed and $1 < c < \min\left\{\dfrac{2\lambda_j + 1 + (\lambda_j + 1)^{\frac{3}{2}}}{\lambda_j}\right\}$, $A_c A_c' - A_d A_d'$ is a positive definite matrix if and only if $0 < d < \min\{(b_2)_j\} < 1$. From Theorem 3.1, the proof for Theorem 4.2(b) is completed. ❑

CONCLUSIONS

A new estimator was introduced from further modification of the Liu-type Estimator. The new estimator, $\tilde{\beta}_c$, was compared with the Ordinary Ridge Regression Estimator, $\hat{\beta}_k$, and the Liu Estimator, $\hat{\beta}_d$, in terms of the mean squared error. The comparison results are presented in Theorem 4.1 and Theorem 4.2. It was found that the accuracy of the new estimator is higher compared to these two estimators because there is a reduction in the mean squared error of the new estimator under certain conditions, which are

(i) $\text{mse}(\hat{\beta}_k) > \text{mse}(\tilde{\beta}_c)$ for $0 < k < \min\{(b_1)_j\}$ and $c > 1$ if $[\text{bias}(\tilde{\beta}_c)]'$ $(A_k A_k' - A_c A_c')^{-1}$ $[\text{bias}(\tilde{\beta}_c)] < \sigma^2$, where $(b_1)_j = \dfrac{\lambda_j (c-1)^2}{\lambda_j^2 + 2c\lambda_j + 2c - 1}$, $j = 1, 2, ..., p$,

(ii) $\text{mse}(\hat{\beta}_d) > \text{mse}(\tilde{\beta}_c)$ for $0 < \max\{(b_2)_j\} < d < 1$ and $1 \leq c \min\left\{\dfrac{2\lambda_j + 1 + (\lambda_j + 1)^{\frac{3}{2}}}{\lambda_j}\right\}$ if $[\text{bias}(\tilde{\beta}_c)]$ $(A_d A_d' - A_c A_c')^{-1} [\text{bias}(\tilde{\beta}_c)] < \sigma^2$, where $(b_2)_j = \dfrac{\lambda_j^2 + 4\lambda_j c - \lambda_j + 2c - 1 - \lambda_j c^2}{(\lambda_j + c)^2}$, $j = 1, 2, ..., p$.

Therefore, this new estimator can be considered as an alternative to estimate the unknown parameter in linear regression models. This new estimator could be recommended to those working on applications involving regression analysis in any field of study such as econometrics, oceanography and geophysics, In directly, the results of the regression analysis could be improved.

This study focused on the estimator for the linear regression model. Extending from this study, future research could be done on exploring the estimators for the non-linear regression model.

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